Holographic description of gravitational anomalies

Sergey N. Solodukhin

School of Engineering and Science, International University Bremen
P.O. Box 750561, Bremen 28759, Germany
E-mail: s.solodukhin@iu-bremen.de

ABSTRACT: The holographic duality can be extended to include quantum theories with the broken coordinate invariance leading to the appearance of the gravitational anomalies. On the gravity side one adds the gravitational Chern-Simons term to the bulk action which is gauge invariant only up to the boundary terms. We analyze in detail how the gravitational anomalies originate from the modified Einstein equations in the bulk. As a side observation, we find that the gravitational Chern-Simons functional has the interesting conformal properties. It is invariant under the conformal transformations. Moreover, its metric variation produces a conformal tensor which is a generalization of the Cotton tensor to dimension $d+1 = 4k-1$, $k \in Z$. We calculate the modification of the holographic stress-energy tensor that is due to the Chern-Simons term and use the bulk Einstein equations to find its divergence and thus reproduce the gravitational anomaly. The explicit calculation of the anomaly is carried out in dimensions $d = 2$ and $d = 6$. The result of the holographic calculation is compared with that of the descent method and an agreement is found. The gravitational Chern-Simons term originates by the Kaluza-Klein mechanism from a one-loop modification of M-theory action. This modification is discussed in the context of the gravitational anomaly in the six-dimensional $(2,0)$ theory. The agreement with the earlier conjectured anomaly is found.

KEYWORDS: AdS-CFT Correspondence, Black Holes in String Theory, M-Theory.

Contents

1. Introduction

Dualities play an important role in the theoretical concepts of modern physics. In particular, they help to understand the behavior of certain systems at the strong coupling by relating it to the behavior in the weak coupling regime. The AdS/CFT correspondence [1 – 3] (for review see [4] and more recent [5]) is the duality of this sort. Quite remarkably, it relates not only the different regimes but also apparently different theories. On one side of the duality one has superstring theory or M-theory, semiclassically described by 11-dimensional supergravity, on the product of AdS_{d+1} and a compact manifold. On the other side it is the large N quantum strongly interacting conformal theory living on the conformal boundary of the Anti-de Sitter space. The duality works both ways. It can be used to understand the strongly coupled quantum system in terms of the semiclassical gravitational physics in the bulk. On the other hand, the wisdom gained in the long-time study of the quantum non-gravitational models can be directed to solving the long-standing puzzles of the semiclassically quantized gravity. Among such puzzles one finds the problem of the black hole entropy and the unitarity problem.

The duality in question has the interesting geometric aspects. As is known since the earlier works [6] and [7], there is conformal structure associated with infinity of anti-de Sitter space. Namely, one finds that the asymptotic symmetries which preserve the AdS structure also generate the conformal transformations on the boundary at infinity. On the other hand, the boundary metric serves as the Dirichlet data for the boundary value problem associated with the bulk Einstein equations. Solution to this problem is a bulk metric determined by the boundary data. This is one of the reasons why this duality is associated with holography [8, 9]. The latter states, quite generally, that the fundamental

degrees of freedom are that of the boundary and predicts the possibility to project the bulk physics to the boundary. In the Maldacena's picture the semiclassical gravitational action in the bulk becomes the quantum generating functional for the theory on the boundary. In particular, its variation with respect to the boundary metric (which is considered as the source for the dual stress tensor) produces the n-point correlation functions of the stress-energy tensor in the boundary theory. The one-point function determines, for instance, the conformal anomaly in the boundary theory. Thus, at least in principle, the classical geometry of an asymptotically AdS space provides us with a complete solution to the quantum dual theory. The bulk action has infra-red divergences since it involves the integration over an infinite volume. These are the UV divergences on the boundary theory side. Thus, for this procedure to work the action should be properly regularized by adding the suitable counterterms. These and other questions were actively studied in the literature, see [10]–[24]. The mathematical side of the story was reviewed in [25].

This line of research turned recently to a new interesting direction related to the possibility to understand holographically the gravitational anomalies which may arise in the dual theory [26, 27]. Indeed, the dual theory is generically chiral. The quantization of such a theory may break the coordinate invariance and lead to the anomalies. These anomalies are well studied [28, 29] and are known to appear in dimension $d = 4k-2$, $k \in Z$. In two dimensions they arise in a theory in which the left and right central charges are not equal. In six dimensions the gravitational anomaly arises, in particular, in the (2,0) theory. In the weak coupling regime this theory is described by certain tensor multiplet theory while in the other regime the strongly interacting $(2,0)$ theory describes N coincident M5 branes. Holographically, the gravitational anomaly originates from the gravitational Chern-Simons term[1] which can be added to the gravitational bulk action. This term is not gauge invariant, the non-invariance resides on the boundary that is the source for the anomaly in the boundary theory. In fact, this mechanism is similar to the known [3] holographic origin of the gauge field anomaly that relates it to the appearance of the gauge field Chern-Simons action in the bulk. In this case the Chern-Simons term is related by supersymmetry to the Einstein-Hilbert action and, thus, appears in the leading order in N. On the other hand, the gravitational Chern-Simons term may originate by Kaluza-Klein mechanism from a one-loop modification of the M-theory action [30]. Thus, the gravitational anomaly appears in the subleading order in N. It should be said that the quantum anomalies are important, and sometimes the only one available, source of information about the strongly coupled theory. That is why they should be paid our special attention.

In this paper we give an exhaustive analysis of the holographic gravitational anomaly. Since there is no literature on the gravitational Chern-Simons terms beyond 3 dimensions we start with a detail study of their general properties. In particular, we observe that these are conformally invariant functionals. The field equations which follow from the Chern-

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[1]By the gravitational CS term we mean the term for the Lorentz group $SO(d + 1)$ defined with respect to the spin connection which is on the other hand is completely defined in terms of the vielbein. This is different from the Chern-Simons term for the AdS group $SO(d + 1, 1)$ considered in the context of AdS/CFT correspondence in [23]. The CS term in this case is polynomial in curvature and does not lead to the appearance of the gravitational anomalies in the boundary theory.

Simons action are, thus, traceless. This is not all, however, to the conformal properties. The metric variation of the Chern-Simons term results in a conformal tensor. This means that under the conformal transformations it rescales by a scalar factor. The conformal tensors which we have found exist in any dimension $d+1 = 4k-1$, $k \in Z$ and are different from the known Weyl tensors. Since conformal tensors play an important and special role both in physics and mathematics it would be interesting to see if the tensors we have discovered have their place in the available list of conformal tensors.

The modified Einstein equations in the bulk are then subject to the Dirichlet problem. We fix the boundary metric and solve the equations by doing the Fefferman-Graham expansion for the bulk metric. The full analysis of the problem is rather complicated. However, in order to gain information about the divergence of the dual stress tensor we have to look at certain (dependent on the boundary dimension d) order in the expansion of (r, i) component of the Einstein equations. This way we calculate the gravitational anomaly in $d = 2$ and $d = 6$. The holographic stress-energy tensor is defined conventionally as a variation of the gravitational action with respect to the boundary metric. We carry out the calculation of the stress-energy tensor that is due to the presence of the Chern-Simons term in the bulk action and find a general expression for the tensor that is valid in any dimension d. We then compare the holographic anomalies with what one obtains in the standard descent method and find agreement. Finally, we analyze the gravitational anomaly in six dimensions as arising holographically from a one-loop modification of the gravitational action. We compare it with the conjectured anomaly for the $(2, 0)$ theory and find a complete agreement. Before turning to the analysis let us emphasize that throughout the paper we consider the space-time of Euclidean signature and use the standard (see [31] for instance) conventions for the definition of the curvature.

2. Brief review of gravitational anomalies

In this section our main source is the original paper [28] and the second volume of the book [32]. A more recent review is [33]. In the parity-preserving case one can always employ the Pauli-Villars regularization of loop diagram that preserves the gauge invariance. The violation of gauge invariance occurs for fields whose gauge couplings violate parity. This is the case for the general coordinate invariance for fields which are in a complex (or pseudo-real) representation of the Lorentz group that violates parity. In Euclidean signature, the complex representations of the Lorentz group $SO(d)$ of d-dimensional Minkowski space occurs if dimension $d = 4k - 2$.

As other gauge anomalies, the gravitational anomaly has a topological origin and is related to certain topological invariants of the tangent bundle in dimension $d + 2 = 4k$. These invariants are polynomial in the Riemann curvature $R^a_b = \frac{1}{2} R^a_{b\mu\nu} dx^\mu \wedge dx^\nu$ two-form. We remind that $R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$, where $\omega^a_b = \omega^a_{b,\mu} dx^\mu$ is the spin connection one-form, and, with respect to indices a and b, is antisymmetric $d \times d$ matrix. Important property is that the trace of any odd number of matrices R^a_b vanishes,

$$\text{Tr}(R^{2k-1}) = 0 \ \ .$$

Thus only the even powers of R can be used to construct the invariants. Since the latter should be further integrated over a manifold M of dimension D, only if $D = 4n$ such invariants are non-trivial. The gravitational anomaly in dimension d is obtained by the descent mechanism from the invariants in two dimensions higher, $D = d + 2$. This is yet another reason why the gravitational anomaly should be expected to appear in the dimension $d = 4k - 2$.

There are certain combinations of invariants constructed from R which are sort of primary and are called the Pontryagin classes. Notice, that if d is even, by an orthogonal transformation such an antisymmetric $d \times d$ matrix can be brought to a skew diagonal form in terms of its eigen-values x_i, $i = 1 \ldots \frac{d}{2}$. The characteristic Pontryagin class $p_i(M)$ is defined by

$$
\begin{aligned}
\det\left(1 - \frac{1}{2\pi}R\right) &= \sum_{k=0}^{\infty} \frac{p_k}{(2\pi)^{2k}} \\
p_0(M) &= 1 \\
p_1(M) &\equiv \sum_i x_i^2 = -\frac{1}{2}\operatorname{Tr}R^2 \\
p_2(M) &\equiv \sum_{i<j} x_i^2 x_j^2 = -\frac{1}{4}\operatorname{Tr}R^4 + \frac{1}{8}(\operatorname{Tr}R^2)^2 \\
p_3(M) &\equiv \sum_{i<j<k} x_i^2 x_j^2 x_k^2 = -\frac{1}{6}\operatorname{Tr}R^6 + \frac{1}{8}\operatorname{Tr}R^2\ \operatorname{Tr}R^4 - \frac{1}{48}(\operatorname{Tr}R^2)^3 \ . \quad (2.1)
\end{aligned}
$$

In the descent mechanism, just mentioned, one substitutes R_{ab} in (2.1) by $R'_{ab} = R_{ab} + \nabla_a \xi_b - \nabla_b \xi_a$ and then expands everything to the first order in ξ_a. Being integrated over a d-dimensional manifold the result takes the general form

$$
\int d^d x \xi^\mu X_\mu \ ,
$$

where X_μ is constructed via the Riemann tensor and its first derivative, thus leading to the anomaly in the non-conservation of the stress-energy tensor

$$
\nabla_\alpha T^\alpha_\mu = X_\mu \ . \quad (2.2)
$$

The concrete form of X_μ depends on the dimension $d = 4k - 2$ and the type of the field. The anomaly for spin 1/2 particle is determined by this mechanism applied to the Dirac genus[2]

$$
\hat{I}_{1/2} = \prod_i \frac{x_i/2}{\sinh(x_i/2)} \ .
$$

Its expansion in terms of the eigenvalues x_i gives

$$
\hat{I}_{1/2} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \frac{1}{2^6 15120}(-16p_3 + 44p_1 p_2 - 31p_1^3) + \cdots \quad (2.3)
$$

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[2]In order to remove a common factor one usually defines $I_{1/2} = -i(2\pi)^{-D/2}\hat{I}_{1/2}$ with similar definitions of $\hat{I}_A$.

The anomaly for antisymmetric self-dual tensor is described by

$$\hat{I}_A = -\frac{1}{8} \prod_i \frac{x_i}{\tanh x_i} \tag{2.4}$$

which has expansion

$$\hat{I}_A = -\frac{1}{8} - \frac{p_1}{24} + \frac{1}{5760}(16p_1^2 - 112p_2) + \frac{1}{967680}(-7936p_3 + 1664p_1p_2 - 256p_1^3) + \cdots \tag{2.5}$$

Another field which may contribute to the gravitational anomaly is gravitino. Its anomaly is determined by the corresponding invariant polynomial. Since the theory on the boundary of AdS is not supposed to contain gravity we skip the discussion of the anomaly due to gravitino.

In addition to the pure gravitational anomalies there may be the mixed anomalies which are due to loop diagrams that contain both external gravitons and gauge fields. Thus, the chiral field should carry the Yang-Mills charge. The only massless chiral field of this type is Weyl spinor. The mixed anomaly then is determined by invariant polynomials involving both the curvature R two-form and the field strength $F = dA + A \wedge A$ of the Yang-Mills field. For gauge field in real representation of the gauge group we have that $\operatorname{tr} F^{2k+1} = 0$ and the relevant polynomial is

$$\hat{I}_{1/2}(F, R) = \operatorname{tr}(\cos F)\hat{I}_{1/2}(R) \ , \tag{2.6}$$

where $\hat{I}_{1/2}(R)$ was introduced above. It has the following expansion

$$\hat{I}_{1/2}(F, R) = n + \left[c_2 - \frac{n}{24}p_1\right] + \left[-\frac{1}{6}\left(c_4 + \frac{1}{2}c_2^2\right) + \frac{n}{5760}(7p_1^2 - 4p_2) - \frac{p_1}{24}c_2\right] + \cdots \ , \tag{2.7}$$

where $n = \operatorname{tr} 1$ is dimension of the representation of the gauge group and $c_j(F)$ is the Chern class defined as $\det(1 + iF/2\pi) = \sum_j i^j c_j(F)/(2\pi)^j$. In terms of the field strength we have that $c_0(F) = 1$, $c_2(F) = -\frac{1}{2}\operatorname{tr} F^2$, $c_4(F) = \frac{1}{8}(\operatorname{tr} F^2)^2 - \frac{1}{4}\operatorname{tr} F^4$.

3. Gravitational Chern-Simons terms

The gravitational Chern-Simons terms Ω_{2n+1} are defined as[3]

$$d\Omega_{2n+1} = \operatorname{Tr} R^{n+1} \tag{3.1}$$

and are certain polynomials of the spin connection $\omega^a_{\ b}$ and its exterior derivative $d\omega^a_{\ b}$ (or, equivalently, of curvature $R^a_{\ b}$). The closed form for arbitrary n is

$$\Omega_{2n+1} = (n + 1) \int_0^1 dt \ t^n \operatorname{Tr}(\omega(d\omega + t\omega^2)^n) \ . \tag{3.2}$$

Both the spin connection $\omega^a_{\ b}$ and the curvature $R^a_{\ b}$ take values in the algebra of the Lorentz group so that the other name for Ω_{2n+1} is the Lorentz Chern-Simons term. Thus, both ω

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[3]We will skip writing symbol $\wedge$ for the wedge product of several differential forms.

and R are antisymmetric in the Lorentz indices. Variation of the term (3.2) under a small change of the spin connection is

$$\delta\Omega_{2n+1} = (n+1)\operatorname{Tr}(\delta\omega R^n) + d(\ldots) \ , \qquad (3.3)$$

where $d(\ldots)$ stands for a term which is exact form. As was discussed in section 2, R is antisymmetric matrix so that the trace of the product of odd number of R gives zero. Similarly, we have that $\operatorname{Tr}(\delta\omega R^{2k}) = 0$ that can be shown by taking the transposition of this expression. Thus, for even n the right hand side of both (3.1) and (3.3) is vanishing (in the case of (3.3) it is up to an exact form). So that, neither action $\int \Omega_{2n+1}$ produces the non-trivial field equations if n is even. The case of odd $n = 2k-1$ will be further considered. The Chern-Simons action

$$W_{\mathsf{CS}} = a_n \int_{M^{2n+1}} \Omega_{2n+1} \ , \qquad a_n = \frac{2^n}{n+1} \ , \qquad (3.4)$$

where $n = 2k-1$ with integer k, describes non-trivial dynamics for the gravitational field.

The spin connection is not independent variable. It is determined by equation

$$de^a + \omega^a{}_b \wedge e^b = 0 \ , \qquad (3.5)$$

where $e^a = h^a_\mu dx^\mu$ is the vielbein, a "square root" of metric, $G_{\mu\nu} = h^a_\mu h^b_\nu \delta_{ab}$. The components of the vielbein can be used to project the local Lorentz indices to the coordinate indices and vice versa. Useful formula for calculating the components of the spin connection in terms of the vielbein is

$$\omega_{ab,\mu} = \frac{1}{2}(C_{a\nu\mu} h^\nu_b + C_{b\mu\nu} h^\nu_a - C_{d\alpha\beta} h^\alpha_a h^\beta_b h^d_\mu) \ ,$$
$$C^a_{\mu\nu} \equiv \partial_\mu h^a_\nu - \partial_\nu h^a_\mu \ . \qquad (3.6)$$

The Riemann curvature satisfies the two types of identities

$$\begin{aligned} R^a{}_{[\mu,\alpha\beta]} = 0 &\iff R^a{}_b \wedge e^b = 0 \quad (1) \\ \nabla_{[\alpha} R^{\mu\nu}{}_{\beta\gamma]} = 0 &\iff \nabla R^a{}_b = 0 \quad (2) \end{aligned} \qquad (3.7)$$

which will be useful in our analysis.

Conformal invariance. In this section we would like to find the general form for the field equations which follow from the Chern-Simons action (3.4) when we vary the vielbein. This will be done in a moment. We pause here to show that the gravitational Chern-Simons is actually conformal invariant so that the field equation which follows from (3.4) should be traceless. It immediately follows from (3.6) that under the rescaling of the vielbein, $h^a_\mu \to e^\sigma h^a_\mu$, the components of spin connection change as

$$\omega_{ab,\mu} \to \omega_{ab,\mu} + \partial_b \sigma h_{a\mu} - \partial_a \sigma h_{b\mu} \ , \qquad (3.8)$$

where we define $\partial_a \equiv h^\mu_a \partial_\mu$. The conformal variation of the bulk part of the Chern-Simons term (3.3) vanishes due to the Bianchi (1) identity. The action (3.4) is thus conformal invariant provided that the conformal parameter σ vanishes on the boundary of M^{4k-1}. This is an interesting feature common to the gravitational Chern-Simons terms in all dimensions.

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Field equations. Now we are in a position to find an explicit form for the field equations which follow from the Chern-Simons action (3.4) when we vary the vielbein h^a_μ. We first rewrite the integrated variation formula (3.3) in components and neglect possible boundary terms[4]

$$\delta W_{\text{CS}} = \int d^{2n+1}x \; h \; \epsilon^{\sigma_1\sigma_2...\sigma_{2n}\mu} \; R^a{}_{a_1\sigma_1\sigma_2} R^{a_1}{}_{a_2\sigma_3\sigma_4} \cdots \; R^{a_{2n-2}}{}_{b \; \sigma_{2n-1}\sigma_{2n}} \delta\omega^b{}_{a,\mu} \; , \qquad (3.9)$$

where $h = \det h^a_\mu$. Variation of the spin connection (3.6) under the infinitesimal change of the vielbein

$$\delta\omega^a{}_{b,\mu} = \delta\Gamma^\alpha_{\mu\nu} h^a_\alpha h^\nu_b - h^\nu_b \nabla_\mu \delta h^a_\nu \qquad (3.10)$$

is a combination of a part due to variation of the vielbein alone and of another part which is due to variation of the vielbein inside the metric. The latter comes from the variation of the Christoffel symbol

$$\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2}[-\nabla^\alpha \delta g_{\mu\nu} + \nabla_\mu \delta g^\alpha{}_\nu + \nabla_\nu \delta g^\alpha{}_\mu] \; , \quad \delta g^\alpha{}_\mu \equiv g^{\alpha\nu}\delta g_{\mu\nu} \; . \qquad (3.11)$$

Substituting (3.10) into (3.9) we notice that the part due to the variation of the vielbein vanishes after integrating by parts and using the Bianchi (2) identities. The only non-trivial variation thus comes from that of the metric. This variation of the Chern-Simons term can be shown to vanish (provided that both types of the Bianchi identities are used) identically if n is even. This is of course consistent with the arguments given earlier in this section. If n is odd the variation is non-trivial

$$\delta W_{\text{CS}} = -2 \int_{M^{2n+1}} d^{2n+1}x \; h \; \delta g_{\mu\nu} \; C^{\mu\nu} \qquad (3.12)$$

with a tensor $C^{\mu\nu}$ defined as

$$C^{\mu\nu} = \nabla_\alpha S^{(\mu\nu)\alpha} \; ,$$
$$S^{\mu\nu\alpha} = -\frac{1}{2}\epsilon^{\sigma_1\sigma_2...\sigma_{2n}\mu} \; R^\nu{}_{a_1\sigma_1\sigma_2} R^{a_1}{}_{a_2\sigma_3\sigma_4} \cdots \; R^{a_{2n-2}\alpha}{}_{\sigma_{2n-1}\sigma_{2n}} \; , \qquad (3.13)$$

where symmetrization is defined as $B^{(\mu\nu)} = \frac{1}{2}(B^{\mu\nu} + B^{\nu\mu})$. The tensor $S^{\mu\nu\alpha}$ is antisymmetric in last two indices. It is vanishing when the trace over any pair of indices is taken and is covariantly conserved,

$$S^{\mu\nu\alpha} = -S^{\mu\alpha\nu} \; , \qquad S^{\alpha\nu}{}_\alpha = 0 \, , \qquad \nabla_\mu S^{\mu\nu\alpha} = 0 \; . \qquad (3.14)$$

In this respect it resembles the tensor of spin. We however do not pursue this analogy in the present paper. By virtue of the Bianchi identities (3.7) the tensor $C^{\mu\nu}$ is traceless and covariantly conserved.

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[4]We use that $dx^{\sigma_1} \wedge \ldots \wedge dx^{\sigma_{2n+1}} = \epsilon^{\sigma_1\cdots\sigma_{2n+1}} h d^{2n+1}x$, $h = \det h^a_\mu$ and $\epsilon^{\sigma_1\sigma_2\cdots} = h^{\sigma_1}_{a_1} h^{\sigma_2}_{a_2} \ldots \epsilon^{a_1 a_2 \cdots}$.

Dimension $d + 1 = 3$ ($n = 1$). The three-dimensional General Relativity with the gravitational Chern-Simons term added is known as a topologically massive gravity and was first considered in [34] and [35]. In the three-dimensional case we have that

$$S^{\mu\nu\alpha} = -\frac{1}{2}\epsilon^{\sigma_1\sigma_2\mu}R^{\nu\alpha}_{\sigma_1\sigma_2} \quad . \tag{3.15}$$

This can be further brought to another form using the fact that the Riemann tensor in three dimensions is expressed in terms of the Ricci tensor and the Ricci scalar as follows

$$R^{\nu\alpha}_{\sigma\rho} = \delta^\nu_\sigma P^\alpha_\rho + \delta^\alpha_\rho P^\nu_\sigma - \delta^\nu_\rho P^\alpha_\sigma - \delta^\alpha_\sigma P^\nu_\rho \quad ,$$

where $P^\alpha_\beta = R^\alpha_\beta - \frac{1}{4}\delta^\alpha_\beta R$. By means of this relation we find that

$$S^{\mu\nu\alpha} = \epsilon^{\sigma\nu\mu}P^\alpha_\sigma + \epsilon^{\mu\alpha\sigma}P^\nu_\sigma \quad . \tag{3.16}$$

The first term in the above expression is antisymmetric in μ and ν so it drops out in the symmetrization (3.13). The second term, on the other hand, is symmetric in indices μ and ν that can be shown by contracting this term with $\epsilon_{\mu\nu\rho}$ and demonstrating that this gives zero provided the Bianchi identities are employed once again. We finally have that

$$C^{\mu\nu} = \nabla_\alpha S^{(\mu\nu)\alpha} = \epsilon^{\mu\alpha\sigma}\nabla_\alpha\left(R^\nu_\sigma - \frac{1}{4}\delta^\mu_\sigma R\right) \quad . \tag{3.17}$$

In three dimensions the tensor $C^{\mu\nu}$ is known as the Cotton tensor. It plays an important role since it is the only conformal tensor available in three dimensions. Expression (3.13) gives a generalization[5] of the Cotton tensor to higher dimensions ($n > 1$). The higher dimensional generalization gives the conformal tensors as well.

Conformal property of $C^{\mu\nu}$. The tensor $C^{\mu\nu}$ defined in (3.13) is a conformal tensor of weight $-(d+3)$. This property makes it similar to Weyl tensor. In order to obtain the transformation law for the tensor $C^{\mu\nu}$ in dimension $d+1$ ($d = 2n = 4k-2, k \in Z$) we first note that under the infinitesimal conformal transformation $\delta_\sigma h^a_\mu = \delta\sigma h^a_\mu$ the tensor $S^{(\mu\nu)\alpha}$ transforms as follows

$$\delta_\sigma S^{(\mu\nu)\alpha} = -(d+3)\delta\sigma \, S^{(\mu\nu)\alpha} + \epsilon^{\sigma_1\ldots\sigma_{d-1}\alpha(\mu}R^{\nu)}_{a_1\sigma_1\sigma_2}\cdots R^{a_{n-2}}_{a_{n-1}\sigma_{d-3}\sigma_{d-2}}\nabla_{\sigma_{d-1}}\nabla^{a_{n-1}}\delta\sigma \; . \tag{3.18}$$

From this it is straightforward to derive that

$$\nabla_\alpha\{\delta S^{(\mu\nu)\alpha}\} = -(d+3)\delta\sigma \, S^{(\mu\nu)\alpha} - (d+2)S^{(\mu\nu)\alpha} \, \partial_\alpha\delta\sigma \quad . \tag{3.19}$$

Combining this with the obvious property

$$\delta(\nabla_\alpha)S^{(\mu\nu)\alpha} = (d+2)S^{(\mu\nu)\alpha} \, \partial_\alpha\delta\sigma \tag{3.20}$$

we find that tensor $C^{\mu\nu} = \nabla_\alpha S^{(\mu\nu)\alpha}$ transforms as

$$\delta C^{\mu\nu} = -(d+3)\delta\sigma \, C^{\mu\nu} \tag{3.21}$$

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[5]There have been earlier suggested some generalizations [36] of the Cotton tensor to higher dimensions. These are however linear in the Riemann curvature and thus differ from (3.13).

under the conformal transformations. As is well known (see, for instance, Proposition 2.1 in [40]) the transformation law (3.21) under the infinitesimal conformal transformations implies that the tensor $C_{\mu\nu}$ is conformal and changes properly under the finite conformal transformations. On the other hand, this property follows directly from the fact that $C^{\mu\nu}$ is obtained as the metric variation of a conformal invariant functional (a nice discussion of this general fact can be found, for instance, in [18]). The tensor $C^{\mu\nu}$, thus, vanishes for any metric conformal to the maximally symmetric, constant curvature, metric $g_{\mu\nu}^{\mathsf{cc}}$. Indeed, the Riemann tensor $R^{\alpha\beta}{}_{\mu\nu} = \frac{R}{d(d+1)}(\delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta)$ for a maximally symmetric metric so that the tensor $S^{\mu\nu\alpha}$, and hence $C^{\mu\nu}$, vanishes identically in this case.

Thus, the tensors $C^{\mu\nu}$ (3.13) share same properties in all dimensions $4k - 1$, $k \in Z$: they are traceless, covariantly conserved and conformal. Conformal tensors traditionally play special role in differential geometry and their complete classification is a long-standing problem. We are, however, not aware of any earlier appearance of tensors (3.13) in the mathematics or physics literature.

This tensor, actually, differs from all known conformal tensors in an interesting way. Consider metric $g_{\mu\nu} = g_{\mu\nu}^{\mathsf{cc}} + \eta_{\mu\nu}$ which is a small deformation of a constant curvature maximally symmetric metric $g_{\mu\nu}^{\mathsf{cc}}$. Usually, a conformal tensor T for such a deformation takes the form (skipping the indices) $T = \mathcal{D}\eta$ with $\mathcal{D}$ being some invariant differential operator. Such operators can be classified that allows to classify all conformal invariants which are represented in such a form for a small deformation of the maximally symmetric metric. The corresponding classification theorem is due to Graham and Hirachi [38]. In particular, it says that in odd dimensions (which is of our interest) there is only Weyl tensor. Interestingly, tensor $C^{\mu\nu}$ (3.13) does not fit in the conditions of this theorem[6]. It is polynomial in small deformation of the maximally symmetric metric, $C[g^{\mathsf{cc}} + \eta] \sim \eta^{2k-1}$, in dimension $4k - 1$, $k > 1$. This can be easily seen already for tensor $S^{\mu\nu}$: due to the Bianchi identities the linear term and all terms η^{2l-1}, $l < k$ vanish identically. In dimension 7 one can find explicit form for the leading term. It is more convenient to write it for the tensor $S^{\mu\nu\alpha}$,

$$S^{(\mu\nu)\alpha} = -\frac{1}{2}\epsilon^{\sigma_1...\sigma_6(\mu}D^{\nu)}{}_{\sigma_1 a_1 \sigma_2}D^{a_1}{}_{\sigma_3 a_2 \sigma_4}D^{a_2}{}_{\sigma_5}{}^\alpha{}_{\sigma_6}$$
$$+\frac{R}{2d(d+1)}\epsilon^{\sigma_1...\sigma_5\alpha(\mu}D^{\nu)}{}_{\sigma_1 a_1 \sigma_2}D^{a_1}{}_{\sigma_3 a_2 \sigma_4}\eta^{a_2}{}_{\sigma_5} , \qquad (3.22)$$

where we have introduced notation

$$D^\alpha{}_{\sigma_1 a \sigma_2} = \nabla_{\sigma_1}(\nabla_a \eta^\alpha{}_{\sigma_2} - \nabla^\alpha \eta_{a\sigma_2}) , \quad \eta^\alpha{}_\sigma = g_{\mathsf{cc}}^{\alpha\beta}\eta_{\beta\sigma} .$$

Reducible Chern-Simons terms. So far we have considered the irreducible form of the Chern-Simons terms. There can be, however, forms which reduce to the product of several such terms. An example is

$$W_{\mathsf{CS}}^{(k,p)} = (n+1)a_n \int_{M^{2n+1}} \Omega_{2k+1} d\Omega_{2p+1} , \qquad n = k + p + 1 . \qquad (3.23)$$

---

[6]I thank Robin Graham for discussion on this point.

The metric variation

$$\delta W_{\text{CS}}^{(k,p)} = -8 \int_{M^{2n+1}} C_{(k,p)}^{\mu\nu} \delta g_{\mu\nu} \tag{3.24}$$

of this action gives a tensor

$$C_{(k,p)}^{\mu\nu} = -\frac{1}{8} \epsilon^{\sigma_1 \ldots \sigma_{2n}(\mu} \nabla_\alpha \Big[(k+1) R^{\nu)}{}_{a_1\sigma_1\sigma_2} \ldots R^{a_{2k-2}\alpha}{}_{\sigma_{2k-1}\sigma_{2k}} (R^{c_1}{}_{c_2\sigma_{2k+1}\sigma_{2k+2}} \ldots R^{c_{2p+1}}{}_{c_{2p+2}\sigma_{2n-1}\sigma_{2n}})$$

$$+ (p+1) R^{\nu)}{}_{a_1\sigma_1\sigma_2} \ldots R^{a_{2p-2}\alpha}{}_{\sigma_{2p-1}\sigma_{2p}} (R^{c_1}{}_{c_2\sigma_{2p+1}\sigma_{2p+2}} \ldots R^{c_{2k+1}}{}_{c_{2k+2}\sigma_{2n-1}\sigma_{2n}}) \Big] \ . \tag{3.25}$$

It is traceless and covariantly conserved and is yet another possible generalization of the Cotton tensor to higher dimensions. If one includes the Yang-Mills field into consideration there may appear the mixed terms like

$$W_{\text{mix}} = \int_{M^{2n+1}} \Omega_{2p+1} \operatorname{tr} F^k \ , \quad n = k + p \ . \tag{3.26}$$

The metric variation of this action is obvious.

4. Holographic evaluation of gravitational anomaly

According to the holographic conjecture the (d+1)-dimensional gravitational theory (referred as the bulk theory) is equivalent to a d-dimensional conformal field (boundary) theory. The boundary in question is the boundary of an asymptotically AdS space-time which is a solution to the gravitational bulk theory. More generally, the duality is formulated for string theory (or M-theory) on anti de-Sitter space and the (super)-gravity action is a low-energy approximation to this more fundamental theory. The (super)-gravity action generically has the higher derivative modifications of the purely gravitational part of the action. Here we consider the case when this modification is in the form of the gravitational Chern-Simons terms. These terms may appear in particular due to the Kaluza-Klein reduction of the higher curvature terms generically present in the 11-dimensional M-theory action.

The gravitational theory in (d+1)-dimensional space-time is given by the action

$$W_{\text{gr}} = W_{\text{EH}} - \frac{\beta}{32\pi G_N} W_{\text{CS}} \ , \tag{4.1}$$

which is sum of the Chern-Simons term (3.4) and the ordinary Einstein-Hilbert action (with a negative cosmological constant)

$$W_{\text{EH}} = -\frac{1}{16\pi G_N} \left[\int_{M^{d+1}} (R[G] + d(d-1)/l^2) + \int_{\partial M^{d+1}} 2K \right] \ , \tag{4.2}$$

where K is trace of the second fundamental form of boundary ∂M. G_N is Newton's constant in $d+1$ dimensions. Parameter l sets the AdS scale. We will use units $l = 1$. One can add to the action (4.1) the reducible forms of the gravitational Chern-Simons term existing in the dimension $d + 1$. Note, that the analytic continuation of the Chern-Simons action to Lorentzian signature is somewhat subtle and involves the multiplication by i. So

that if the coupling β is purely imaginary in Euclidean signature (as is reasonable from the boundary point of view since the gravitational anomaly comes from the imaginary part of the quantum action) it becomes real in Lorentzian signature. The analytic continuation of the topological terms is discussed in [37].

The gravitational bulk equations obtained by varying the action (4.1) with respect to the metric takes the form

$$R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R - \frac{d(d-1)}{2}G_{\mu\nu} + \beta C_{\mu\nu} = 0 \; , \tag{4.3}$$

where all curvature tensors are determined with respect to the bulk metric $G_{\mu\nu}$. The tensor $C_{\mu\nu}$ is a result of the variation of the gravitational Chern-Simons term. Although the Chern-Simons terms are defined in terms of the Lorentz connection which is not gauge invariant object the variation is presented in the covariant and gauge invariant form as we have shown in the previous section. This is just a manifestation of the fact that the "non-invariance" of the Chern-Simons term resides on the boundary and does not appear in the bulk field equations. By virtue of the Bianchi identities this quantity (both for the irreducible and reducible Chern-Simons terms) is manifestly traceless and identically covariantly conserved,

$$C_{\mu\nu}G^{\mu\nu} = 0 \,, \qquad \nabla_\mu C^\mu_\nu = 0 \; . \tag{4.4}$$

Due to these properties we find that solution to the equation (4.3) is space-time with constant Ricci scalar $R = -d(d+1)$. This is exactly what we had when the Chern-Simons term was not included in the action. In that case moreover the Ricci tensor was proportional to metric, $R_{\mu\nu} = -dG_{\mu\nu}$. It is no more the case in the presence of the Chern-Simons term and we have

$$R_{\mu\nu} = -dG_{\mu\nu} - \beta C_{\mu\nu} \; . \tag{4.5}$$

This is that equation which we are going to solve. We start with choosing the bulk metric in the form

$$ds^2 = G_{\mu\nu}dX^\mu dX^\nu = dr^2 + g_{ij}(r,x)dx^i dx^j \tag{4.6}$$

that always can be done by using the appropriate normal coordinates. The quantity $g_{ij}(r,x)$ is the induced metric on the hypersurface of a constant value of the radial coordinate r. The following expansion

$$g(r,x) = e^{2r}\left[g_{(0)} + g_{(2)}e^{-2r} + \cdots + g_{(d)}e^{-dr} + h_{(d)}\,re^{-dr} + O(e^{-(d+1)r})\right] \tag{4.7}$$

is assumed so that the metric (4.6) describes an asymptotically anti-de Sitter space-time with $g_{(0)}$ being the metric on its d-dimensional boundary. The non-vanishing term $h_{(d)}$ generically appears in the expansion if dimension d is even. In the mathematics literature this tensor is known as the obstruction tensor (see [38, 39]). It is traceless, covariantly conserved and conformal in any even dimension d. By a general argument given in [16] it is a multiple of the stress tensor derived from the integrated holographic conformal anomaly. It follows immediately that this term vanishes identically when $d = 2$ since the conformal anomaly then is a multiple of the Ricci scalar and, if integrated, gives a topological invariant and no non-trivial metric variation appears.

Holographic stress-energy tensor. The holographic (or dual) stress-energy tensor is generally defined as a variation of the gravitational action with respect to the metric $g_{ij}^{(0)}(x)$ on the boundary. The gravitational action is considered on-shell, i.e. the bulk metric is supposed to solve the Einstein equations subject to the Dirichlet boundary condition. Since the boundary is at infinity the action should be properly defined. A simple way to do it is to consider a sequence of boundaries at finite value of the radial coordinate r with induced boundary metric $g_{ij}(x, r)$ (for large r we know that $g_{ij}(x, r) = e^{2r}g_{(0)ij}(x) + \cdots$). This way we get a regulated gravitational action. However, this action is typically divergent when regulator r is taken to infinity. On the boundary theory side these divergences have a natural interpretation as the UV divergences. Some *renormalization* is typically needed. A rather natural way to *renormalize* the divergences is to add some local boundary counter terms [10, 11, 17, 16] to the action. These boundary terms do not change the bulk field equations. They not just cancel the divergences but also contribute to the finite part of the action and, in particular, to the finite part of the holographic stress-energy tensor. When the boundary dimension d is odd the exact form of the holographic stress tensor is known [16]. It is determined only by the coefficient $g_{ij}^{(d)}(x)$ in the Fefferman-Graham expansion taken with the appropriate coefficient. When the dimension d is even no general form of the dual stress-energy tensor is known except in some particular cases, $d = 2$, $d = 4$ and $d = 6$ [16]. The expression in terms of the extrinsic curvature (instead of the metric) is however available [20].

In the presence of the Chern-Simons term the holographic stress-energy tensor is modified. Surprisingly, we can get a general form of this modification rather explicitly. In order to see this let us remind the basic steps in defining the holographic stress-energy tensor. Let us introduce small parameter $\varepsilon = e^{-2r}$ which determines the location of the regularized boundary with the induced metric $g_{ij}(x, r(\varepsilon))$ which is the same quantity that appears in (4.6). The expectation value of the stress-energy tensor of the dual theory is then given by

$$\langle T_{ij} \rangle = \frac{2}{\sqrt{\det g_{(0)}}} \frac{\delta W_{\mathtt{gr,ren}}}{\delta g_{(0)}^{ij}(x)} = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon^{d/2-1}} T_{ij}[g] \right) \ , \tag{4.8}$$

where

$$T_{ij}[g] = \frac{2}{\sqrt{\det g(x, r(\varepsilon))}} \frac{\delta W_{\mathtt{gr,ren}}}{\delta g^{ij}(x, r(\varepsilon))} \tag{4.9}$$

is the stress tensor of the theory at finite ε. It contains two contributions,

$$T_{ij}[g] = T_{ij}^{\mathtt{reg}} + T_{ij}^{\mathtt{ct}} \ ,$$

where $T_{ij}^{\mathtt{reg}} = -\frac{1}{8\pi G_N}(K_{ij} - K g_{ij})$ comes from the regulated Einstein-Hilbert action and $T_{ij}^{\mathtt{ct}}$ is the contribution of the boundary counterterms. Their role is to cancel the possible divergences in (4.8) when ε is taken to zero.

We now want to apply this prescription and compute the part in the dual stress-energy tensor which is due to the gravitational Chern-Simons term. All we need to do is

take variation of the Chern-Simons action with respect to the induced metric $g^{ij}(x, r(\varepsilon))$ on the regularized boundary and calculate $T_{ij}[g]$. Then insert it into equation (4.8) and take the limit of ε to zero. The equation (3.9) accompanied with (3.10) and (3.11) is a good starting point for the first step. Notice, that in (3.9) the variation with respect to the boundary value of vielbein results in some term T_i^a which however gives zero after the symmetrization, $h_{a(j}T_{i)}^a$, needed to define the metric stress-energy tensor. Thus, the only contribution to the stress-energy tensor comes from the metric variation in (3.11). We find that

$$
\begin{aligned}
\delta W_{\mathtt{CS}}^{\mathtt{reg}} &= -\frac{\beta}{32\pi G_N} \int d^d x \sqrt{g} \, [S^{ijr} - S^{rji} - S^{irj}]\delta g_{ij} \\
&= -\frac{\beta}{16\pi G_N} \int d^d x \sqrt{g} \, S^{(ij)r}\delta g_{ij} \ ,
\end{aligned}
\tag{4.10}
$$

where tensor $S^{\mu\nu\alpha}$ was introduced in (3.13) and, in the second line, we have used its symmetry properties (3.14). This gives us that

$$
T_{ij}[g] = \frac{\beta}{8\pi G_N} S_{(ij)r}(g(x, r(\varepsilon))) \ .
\tag{4.11}
$$

This should then be expanded in the powers of ε and substituted in (4.8). The analysis shows that the leading divergences in the resultant expression vanish either as a result of the symmetrization in indices i and j or due to the Bianchi identities and we are left with a finite expression. This quite remarkable fact (in dimension $d = 2$ this was observed in [26]) means that the dual stress-energy tensor is finite with no need to introduce any new counterterms. The general form for the finite expression can be easily obtained for arbitrary d by using the expansions (B.4), (B.5) and (B.6) of appendix B,

$$
T_{ij}^{\mathtt{CS}} = -\frac{\beta}{8\pi G_N}\epsilon^{k_1 k_2 \dots k_{d-1}}{}_{(i} R_{j)}{}^{n_1}{}_{k_1 k_2} R_{n_1}{}^{n_2}{}_{k_3 k_4} \dots R_{n_{d-3}}{}^{n_{d-2}}{}_{k_{d-3}k_{d-2}} \, g_{n_{d-2}k_d}^{(2)} \ ,
\tag{4.12}
$$

where one uses the metric $g_{(0)ij}$ to compute the components of the Riemann tensor. Thus, we have to know only the coefficients $g_{(0)ij}$ and $g_{(2)ij}$ in the Fefferman-Graham expansion to determine the part in the dual stress-energy tensor which is due to the Chern-Simons term.

Solving the Einstein equations. The expressions for the components of the bulk curvature are given in appendix A. The strategy of solving the Einstein equations is to substitute the expansion (4.7) in the modified Einstein equations (4.5) and expand both sides of the equations in powers of e^{-r}. Equating coefficients at the same order on both sides one gets the recurrent relations between coefficients of the expansion (4.7) which allow one to determine $g_{ij}^{(n)}(x)$ provided coefficients $g_{ij}^{(k)}(x)$, $k < n$ are already known. The only boundary data required for this procedure to work is value of the boundary metric $g_{ij}^{(0)}(x)$ and the value of the coefficient $g_{ij}^{(d)}(x)$ which is ultimately related to the stress-energy tensor of the boundary CFT. Einstein equations impose constraints on its trace and divergence. The latter thus determines the conservation (or non-conservation) of the stress-energy tensor. The constraint on divergence of $g_{ij}^{(d)}$ appears in the e^{-dr} order of the

expansion of (ri) component of the Einstein equations. It is thus suffice for our purposes to look only at this part of the Einstein equations. The expansion for the inverse metric and the Riemann tensor is given in appendix B. Since we have to calculate the expansion of (ri) component of the tensor $C^{\mu\nu}$ we present below the expression for this component in terms of the tensor S^{ijk},

$$C^{ri} = \frac{1}{2}\left\{\nabla_j S^{rij} + \nabla_j S^{irj} + \partial_r S^{rir} + \frac{1}{2}\mathrm{Tr}(g^{-1}g')S^{rir} + \frac{1}{2}(g^{-1}g')^i_j S^{rrj} - \frac{1}{2}g'_{kn}S^{\mathrm{kin}}\right\},$$
(4.13)

where ∇_j is defined with respect to the effective metric $g_{ij}(r,x)$. The further analysis depends on value of dimension d.

Dimension $d = 2$. The case of two-dimensional boundary was considered in [27] and the holographic tensor was found earlier in [26]. Below we present some details of the analysis. In this case the first non-vanishing contribution to components C^{ri} appears in e^{-4r} order. The components of tensor $S^{\mu\nu\alpha}$ are easy to calculate using (3.15) and the expansion (B.4), (B.5), (B.6) of the Riemann tensor. In the leading order one has

$$S^{rij} = \left(-\frac{1}{2}R\epsilon^{ij} + \epsilon^{ki}g^j_{(2)k} - \epsilon^{kj}g^i_{(2)k}\right)e^{-4r} + \cdots$$
(4.14)
$$S^{irj} = \epsilon^{ji}e^{-2r} + O(e^{-6r})$$
$$S^{rrj} = -\epsilon^{kn}\nabla_k g^j_{(2)n}e^{-4r} + \cdots$$
$$S^{\mathrm{kin}} = \epsilon^{kl}\left(\nabla^i g^n_{(2)l} - \nabla^n g^i_{(2)l}\right)e^{-6r} + \cdots$$

So that the leading term in the expansion of the component of the tensor $C^{\mu\nu}$ can be now calculated using (4.13),

$$C_{ri} = g_{ij}C^{rj} = \left\{-\frac{1}{4}\epsilon^j_i\partial_j R + \frac{1}{2}\left(\epsilon^k_i\nabla_j g^j_{(2)k} + \epsilon^{kj}\nabla_j g_{(2)ki}\right)\right\}e^{-2r} + \cdots$$
(4.15)

As we see from (B.5) and (B.2) the (i,r) component of the Ricci tensor has expansion

$$R_{ri} = \left[-\nabla_j g^j_{(2)i} + \partial_j \mathrm{Tr}\, g_{(2)}\right]e^{-2r} + \cdots$$
(4.16)

Looking now at the expansion of (ir) component of the Einstein equations (4.5) we get the constraint on the coefficient $g_{(2)ij}$ that can be presented in the form

$$\nabla_j t^j_i = -\frac{\beta}{4}\epsilon_i{}^j\partial_j R\ ,$$
$$t_{ij} = g_{(2)ij} - g_{(0)}\mathrm{Tr}\, g_{(2)} + \frac{\beta}{2}\left(\epsilon_i{}^k g_{(2)jk} + \epsilon_j{}^k g_{(2)ik}\right)\ .$$
(4.17)

The holographic stress-energy tensor is defined as

$$T_{ij} = \frac{1}{8\pi G_N}t_{ij}\ .$$
(4.18)

The part in the holographic stress tensor which is due to the Chern-Simons term is in agreement with the general expression (4.12). The divergence of (4.18) produces a gravitational anomaly

$$\nabla_j T_i^j = -\frac{\beta}{32\pi G_N} \epsilon_i{}^j \partial_j R \quad . \tag{4.19}$$

This is precisely the anomaly which is expected to appear in two dimensions. It originates from p_1 (see [28, 41]) via the descent mechanism outlined in section 2.

Dimension d=6. The analysis in six-dimensional case is much more laborious. In the absence of the gravitational Chern-Simons term the analysis was done in [16]. The construction of the holographic stress tensor in terms of the coefficients in the expansion of metric is then already non-trivial and the precise prescription is given in [16]. Turning on the Chern-Simons term makes things even more complicated. Fortunately for us we do not need to go to a full analysis of the modified Einstein equations but have to look only at the e^{-6r} order of the (ir) component of the Einstein equations which determines, as was shown in [16], the conservation law for the holographic stress-energy tensor.

The tensor $S^{\mu\nu\alpha}$ in six dimensions takes the form

$$S^{\mu\nu\alpha} = -\frac{1}{2}\epsilon^{\mu_1...\mu_6\mu} R^\nu{}_{a_1\mu_1\mu_2} R^{a_1}{}_{a_2\mu_3\mu_4} R^{a_2\alpha}{}_{\mu_5\mu_6} \quad . \tag{4.20}$$

The expansion (B.4), (B.5) and (B.6) of the Riemann tensor is sufficient for the analysis of the leading behavior of the components of the tensor (4.20). Below we summarize this analysis:

$$S^{rij} = \left\{ -\frac{1}{2}\epsilon^{k_1...k_6} R^i{}_{n_1k_1k_2} R^{n_1}{}_{n_2k_3k_4} R^{n_2j}{}_{k_5k_6} - \epsilon^{k_1...k_5j} R^i{}_{n_1k_1k_2} R^{n_1}{}_{n_2k_3k_4} g^{n_2}_{(2)k_5} \right\} e^{-8r} \ , \tag{4.21}$$

$$S^{irj} = \left\{ \epsilon^{k_1...k_5i} g^{(2)}_{n_1k_1} R^{n_1}{}_{n_2k_2k_3} R^{n_2j}{}_{k_4k_5} + 2\epsilon^{k_1...k_4ji} g^{(2)}_{n_1k_1} R^{n_1n_2}{}_{k_2k_3} g^{(2)}_{n_2k_4} \right.$$
$$\left. +4\epsilon^{k_1...k_4ji} \nabla_{k_1} g^{(2)}_{nk_2} \nabla_{k_4} g^n_{(2)k_3} \right\} e^{-8r} \ , \tag{4.22}$$

$$S^{rir} = \epsilon^{k_1...k_6} R^i{}_{n_1k_1k_2} R^{n_1}{}_{n_2k_3k_4} \nabla_{k_5} g^{n_2}_{(2)k_6} \ e^{-8r} \ , \tag{4.23}$$

$$S^{\text{kin}} = \epsilon^{k_1...k_5k} R^i{}_{n_1k_1k_2} R^{n_1}{}_{n_2k_3k_4} \nabla^n g^{n_2}_{(2)k_5} \ e^{-10r} \ , \tag{4.24}$$

where we keep only the leading terms, components of the Riemann tensor are defined with respect to metric $g_{(0)ij}$. Notice that in (4.24) we have dropped the terms which vanish when the trace $g^{(0)}_{kn} S^{\text{kin}}$ is taken. Such terms appear both in the order e^{-8r} and in the order e^{-10r} and are not shown in (4.24).

The expansion (4.21), (4.22), (4.23) and (4.24) should be now substituted into equation (4.13). After some reshuffle and noticing that quite a few terms vanish due to the Bianchi identities we get a quite simple result

$$C_{ri} = \nabla_j \left\{ -\frac{1}{4}\epsilon^{k_1...k_6} R^i{}_{n_1k_1k_2} R^{n_1}{}_{n_2k_3k_4} R^{n_2j}{}_{k_5k_6} \right.$$
$$\left. +\frac{1}{2}\epsilon^{k_1...k_5i} R^j{}_{n_1k_1k_2} R^{n_1}{}_{n_2k_3k_4} \ g^{n_2}_{(2)k_5} + \frac{1}{2}\epsilon^{k_1...k_5j} R^i{}_{n_1k_1k_2} R^{n_1}{}_{n_2k_3k_4} \ g^{n_2}_{(2)k_5} \right\} e^{-6r} \quad . \tag{4.25}$$

Obviously, the first term in the brackets is antisymmetric in indices i and j while the two other terms form a symmetric tensor. The latter will modify the holographic stress-energy tensor while the first term will produce a gravitational anomaly.

The expansion of the Ricci tensor to the required order was found in [16]. We refer the reader to that paper for the details. The result is

$$R_{ri} = -3\nabla_j \left(g_{(6)} - A_{(6)} + \frac{1}{24}S \right)^j_{\ i} e^{-6r} \ , \tag{4.26}$$

where we focus only on the term of the order e^{-6r}. The tensors $A_{(6)ij}$ and S_{ij} are local covariant functions of the metric $g_{(0)ij}$ and its derivative, exact expressions are rather lengthy and given in paper [16].

Introduce tensor $t_{ij}^{(\beta)}$ as follows

$$t_{ij}^{(\beta)} = g_{(6)ij} - A_{(6)ij} + \frac{1}{24}S_{ij} - \frac{\beta}{3}\epsilon^{k_1...k_5}_{\quad\ (i}R_{j)n_1k_1k_2}R^{n_1}_{\ n_2k_3k_4} \ g^{n_2}_{(2)\ k_5} \ , \tag{4.27}$$

where the normalization has been chosen in agreement with [16]. Note that in the dimension $d > 2$ the coefficient $g_{(2)}$ is a local covariant function of the metric $g_{(0)}$. In particular, for $d = 6$, we have that

$$g_{(2)ij} = -\frac{1}{4}\left(R_{ij} - \frac{1}{10}Rg_{(0)ij} \right) \ . \tag{4.28}$$

This relation remains the same when the Chern-Simons term is added to the bulk equations. The constraint that comes from the (ir)-component of the Einstein equation, $R_{ir}+\beta C_{ir} = 0$, can be now presented in the following form

$$\nabla_j t^j_{(\beta)i} = -\frac{\beta}{12}\epsilon^{k_1...k_6}\nabla_j(R^i_{\ n_1k_1k_2}R^{n_1}_{\ n_2k_3k_4}R^{n_2j}_{\ \ k_5k_6}) \ . \tag{4.29}$$

Obviously, the tensor $t^j_{(\beta)i}$ (4.27) is defined by this equation only up to covariantly conserved term (proportional to $h_{(6)}$). The holographic stress-energy tensor in the absence of the Chern-Simons term, $\beta = 0$ in this case, was defined in [16]. Extending this definition to the present case and taking into account the general expression (4.12) for the CS contribution we define the stress-energy tensor as follows

$$T_{ij} = \frac{3}{8\pi G_N}t_{ij}^{(\beta)} \ . \tag{4.30}$$

Defined this way this tensor (in the case when $\beta = 0$) was shown in [16] to be symmetric, covariantly conserved and its trace to be the conformal anomaly of the boundary CFT. Notice, that the β-dependent modification in (4.27) is traceless so that the trace of the modified stress-energy tensor remains the same. The stress tensor is however not conserved anymore due to the gravitational anomaly,

$$\nabla_j T_i^j = -\frac{\beta}{32\pi G_N}\epsilon^{k_1...k_6}\nabla_j(R^i_{\ n_1k_1k_2}R^{n_1}_{\ n_2k_3k_4}R^{n_2j}_{\ \ k_5k_6}) \ . \tag{4.31}$$

This is exactly the anomaly that originates in the descent mechanism from the term $\mathrm{Tr}\, R^4$ in the Pontryagin class p_2. This is however not the most general form of the gravitational anomaly in six dimensions. Indeed, another possible anomaly originates from the term $(\mathrm{tr}\, R^2)^2$ which appears both in p_2 and in p_1^2. This anomaly comes out holographically if one adds a reducible form of Chern-Simons term to the bulk gravitational action.

Anomaly from the reducible Chern-Simons term. In six dimensions the only possible reducible form of the Chern-Simons term is $W_{\mathsf{CS}}^{(1,1)} = 4a_3 \int \Omega_3 \wedge d\Omega_3$. Adding this term to the gravitational action

$$W_{\mathrm{gr}} = W_{\mathrm{EH}} - \frac{\beta}{32\pi G_N} W_{\mathsf{CS}} - \frac{\beta_1}{128\pi G_N} W_{\mathsf{CS}}^{(1,1)} \tag{4.32}$$

with some coupling β_1 we get, after some regrouping the terms, the modified Einstein equations

$$R_{\mu\nu} = -dG_{\mu\nu} - \beta C_{\mu\nu} - \beta_1 C_{\mu\nu}^{(1,1)} \ , \tag{4.33}$$

where we took into account that $C_{\mu\nu}^{(1,1)}$ is traceless. Tensor $C_{\mu\nu}^{(1,1)}$ was defined in (3.25) to take the form

$$C_{(1,1)}^{\mu\nu} = -\frac{1}{2}\epsilon^{\sigma_1 \ldots \sigma_6 (\mu} \nabla_\alpha \left[R^{\nu)\alpha}{}_{\sigma_1 \sigma_2} (R^{c_1}{}_{c_2 \sigma_3 \sigma_4} R^{c_2}{}_{c_1 \sigma_5 \sigma_6}) \right] \ , \tag{4.34}$$

where all indices run from 1 to 7. Again we have to look at the (r,i) component of the modified Einstein equations (4.33). The analysis goes through same steps as before, now for the tensor $C_{(1,1)}^{\mu\nu}$. Skipping the details which are pretty straightforward we present the result for the leading term in the large r expansion

$$C_{(1,1)}^{ri} = \nabla_j \left\{ \left[-\frac{1}{4}\epsilon^{k_1 \ldots k_6} R^{ij}{}_{k_1 k_2} + \epsilon^{k_2 \ldots k_6 (i} g_{(2)k_2}^{j)} \right] R^{n_1}{}_{n_2 k_3 k_4} R^{n_2}{}_{n_1 k_5 k_6} \right\} e^{-8r} \ . \tag{4.35}$$

Here all indices (including n_1 and n_2) run from 1 to 6. The new constraint which comes from the (r,i) component of equations (4.33) can be properly formulated in terms of the tensor

$$t_{(\beta,\beta_1)}^{ij} = t_{(\beta)}^{ij} - \frac{\beta_1}{3} \epsilon^{k_2 \ldots k_6 (i} g_{(2)k_2}^{j)} R^{n_1}{}_{n_2 k_3 k_4} R^{n_2}{}_{n_1 k_5 k_6} \ , \tag{4.36}$$

where in the last term the symmetrization in indices i and j is assumed. We can now define the holographic stress tensor as

$$T_{ij} = \frac{3}{8\pi G_N} t_{ij}^{(\beta,\beta_1)} \tag{4.37}$$

in analogy with (4.30). Its divergence is now a combination of the contributions from both the reducible and irreducible Chern-Simons terms

$$\nabla_j T^j{}_i = -\frac{\beta}{32\pi G_N} \epsilon^{k_1 \ldots k_6} \nabla_j (R^i{}_{n_1 k_1 k_2} R^{n_1}{}_{n_2 k_3 k_4} R^{n_2 j}{}_{k_5 k_6})$$
$$\qquad -\frac{\beta_1}{32\pi G_N} \epsilon^{k_1 \ldots k_6} \nabla_j (R^{ij}{}_{k_1 k_2} R^{n_1}{}_{n_2 k_3 k_4} R^{n_2}{}_{n_1 k_5 k_6}) \ . \tag{4.38}$$

The second term in the right hand side of (4.38) is precisely a contribution to the gravitational anomaly from the term $(\mathrm{Tr}\, R^2)^2$ via the descent method. So that equation (4.38) presents the most general form of the gravitational anomaly in six dimensions.

Some comments. The tensor (4.30), or more generally (4.37), has a dual meaning. It is the expectation value of the quantum stress-energy tensor in the dual CFT and is the quasi-local stress-energy tensor introduced by York and Brown [42] to define the energy and angular momentum for solution to the bulk gravitational equations. In three dimensions the Cotton tensor vanishes for any metric conformal to the constant curvature metric. That's why the BTZ metric describing three-dimensional black hole remains a solution to the modified Einstein equations (4.5). The stress-energy tensor (4.18), (4.17) then can be used to calculate the modified values for the mass and angular momentum of the BTZ black hole [26, 27]. In higher dimensions general solution to Einstein equations with a cosmological term is no more maximally symmetric metric so that the tensor $C_{\mu\nu}$ is non-vanishing. This means that some modification of the known solutions describing black hole in asymptotically AdS space-time should be expected. The finding exact solutions to the modified Einstein equations (4.5) or (4.33) is an interesting problem that possibly can be approached numerically. Provided such a solution is known our formulas (4.18), (4.17) or (4.37), (4.36) can be used to calculate the conserved quantities of the solution. We however note that unlike the three-dimensional case in higher dimensions the β-dependent modification in (4.36) vanishes if the boundary metric $g_{ij}^{(0)}(x)$ is flat or is a maximally symmetric constant curvature metric. Only if there is a solution which approaches a non-maximally symmetric metric at infinity then the modification (4.36) or (4.27) of the stress-energy tensor would be relevant. Also, only in this case the gravitational anomaly (4.38) will be actually visible.

5. Remarks on anomalies

Comparison with the descent method. The Chern-Simons term which was added to the bulk gravitational action can be used for calculation of the anomaly using the descent method. In this subsection we do this calculation and compare the resultant anomaly with the one obtained holographically and find that these two anomalies are identical. It is more convenient to calculate first the local Lorentz anomaly and then transform the result to the gravitational anomaly.

We start with some general remarks on the Lorentz symmetry and the Lorentz anomaly. We introduce the vielbein stress-energy action as $T_{(h)a}^i = \frac{2}{h}\frac{\delta W}{\delta h_i^a}$. The subscript (h) is supposed to differ this from the metric stress tensor $T_{(g)}^{ij} = \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{ij}}$. W is the action of the theory in question. These two objects are related as

$$T_{(h)}^{ia} = T_{(g)}^{ij} h_j^a + T_{(g)}^{ji} h_j^a .$$

We raise the Lorentz indices with the help of δ^{ab}. Under the infinitesimal local Lorentz transformations the vielbein and the spin connection transform as

$$\delta h_i^a = \alpha^a{}_b h_i^b, \qquad \delta\omega^a{}_{b,i} = -\partial_i\alpha^a{}_b, \qquad \alpha_{ab} = -\alpha_{ba} . \tag{5.1}$$

In the Lorentz invariant theory one has that $T_{(h)}^{[ab]} = 0$, $T_{(h)}^{ab} = T_{(h)}^{ia} h_i^b$. On quantum level in d-dimensional chiral theory the Lorentz symmetry may be violated if $d = 4k - 2$. In the

descent method the violation is determined by a $(d + 2)$-dimensional invariant form I_{d+2} which is polynomial in the Riemann curvature as was explained in section 2. This form is locally exact $I_{d+2} = dI_{\mathsf{CS}}$, where $I_{\mathsf{CS}}^{(d+1)}$ is a $(d+1)$-dimensional Chern-Simons term. Under the local Lorentz transformations (5.1) this term changes as $\delta_\alpha I_{\mathsf{CS}}^{(d+1)} = d[X^{ab}\alpha_{ba}]$, where $X^{ab} = X^{ab}_{i_1...i_d} dx^{i_1} \wedge \ldots \wedge dx^{i_d}$ is a d-form. The anomaly then shows up in the non-vanishing antisymmetric part of $T^{ab}_{(h)}$ and reads

$$\frac{1}{2}T^{[ab]}_{(h)} = \epsilon^{i_1...i_d} X^{ab}_{i_1...i_d} \quad . \tag{5.2}$$

For instance, take $I_{\mathsf{CS}}^{(d)} = -\frac{\beta}{32\pi G_N} a_n \Omega_{\mathsf{CS}}^{(2n+1)}$ and apply the descent procedure. Using (3.9) we get that it leads to the Lorentz anomaly

$$\frac{1}{2}T^{[ab]}_{(h)} = -\frac{\beta}{32\pi G_N}\epsilon^{i_1...i_d}\left(R^a_{\ c_1,i_1 i_2}\ldots R^{bc_{d-2}}_{\quad i_{d-1}i_d}\right) \quad , \tag{5.3}$$

the expression in the right hand side is obviously antisymmetric in indices a and b if $d = 4k - 2$. The non-vanishing antisymmetric part of $T^{ab}_{(h)}$ would imply that the metric stress-energy tensor is not symmetric. Keeping $T^{ij}_{(g)}$ symmetric we should subtract the antisymmetric part $\frac{1}{2}h^i_a h^j_b T^{[ab]}_{(h)}$. The resultant symmetric tensor is not conserved,

$$\nabla_j T^{ij}_{(g)} = -\frac{\beta}{32\pi G_N}\epsilon^{i_1...i_d}\nabla_j\left(R^i_{\ c_1,i_1 i_2}\ldots R^{jc_{d-2}}_{\quad i_{d-1}i_d}\right) \quad . \tag{5.4}$$

This result is the same as if we replaced $R_{ab} \to R_{ab} + 2\nabla_{[a}\xi_{b]}$ in I_{d+2} and looked at the first order in ξ_a term. This latter prescription was given in section 2. Comparing (5.4) to the holographic expressions (4.19) ($d = 2$) and (4.31) ($d = 6$) we see that in two different methods, by adding the Chern-Simons action $\int I_{\mathsf{CS}}^{(d+1)}$ to the bulk gravitational action and looking at the divergence of the dual stress tensor in the holographic method and, in the second method, by using the same form $I_{\mathsf{CS}}^{(d+1)}$ in the descent procedure, we get same result. Same is true for the anomaly determined by the reducible form (verified for $\Omega_3 d\Omega_3$ when $d = 7$) of the Chern-Simons action. Here we have checked this by brute force. However, it seems that there may be a more general proof that two methods lead to same result[7]. It would be interesting to understand this issue.

Anomaly in $(2, 0)$ six-dimensional conformal theories. In six dimensions there are two known $(2, 0)$ supersymmetric conformal theories. The first one is the free tensor multiplet theory which describes the low energy dynamics of a single M5 brane. The other one is the strongly interacting $(2, 0)$ conformal theory describing N coincident M5 branes. Some information about this second theory can be gained from its conjectured holographic duality to M-theory (or, in large N, limit to the 11-dimensional supergravity) on $AdS_7 \times S^4$ background. In particular, the holographic anomalies proved to be an important source of information about the theory. The conformal anomaly in the $(2, 0)$ theory was calculated in [10]. The comparison to the anomaly in the free tensor multiplet was done for instance

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[7]I thank Jan de Boer and Kostas Skenderis for suggesting this to me.

in [43]. The conformal anomaly in two theories are mainly related by factor $4N^3$. This is the leading contribution to the anomaly which holographically originates from the tree level supergravity action linear in the curvature. The one-loop effective action contains quartic in curvature terms. They lead to the $O(N)$ modification of the anomaly. A nice discussion of this can be found in [45].

The maximal $(2,0)$ supersymmetric theories are necessarily chiral so that the gravitational anomaly is expected to appear. The free tensor multiplet consists of 5 scalars, a (anti)selfdual antisymmetric tensor and 2 Weyl fermions. The gravitational anomaly is thus a descent of 8-form

$$I_8^{\texttt{tens}} = I_A + 2I_{1/2} = -\frac{i}{(2\pi)^3 192}\left[\operatorname{Tr} R^4 - \frac{1}{4}(\operatorname{Tr} R^2)^2\right] \ , \tag{5.5}$$

where we use formulas of section 2. The corresponding anomaly of interacting $(2,0)$, as conjectured in [44] (by assuming that the M5-brane anomaly should be compensated by the inflow anomaly), is determined by

$$I_8^{(2,0)} = N I_8^{\texttt{tens}} \ . \tag{5.6}$$

Note that we focus on the gravitational part of the anomaly neglecting the gauge field and the mixed anomalies.

The anomaly (5.6) is subleading in N which means that holographically it originates from a one-loop term in the effective action. Terms of this type were studied in [45]. There are few terms in the one-loop action which are quartic in curvature. The one of our interest contains invariant $\operatorname{Tr} R^4 - \frac{1}{4}(\operatorname{Tr} R^2)^2$ which is exactly of the type that appears in (5.6), (5.5). More precisely one finds (we use notations of [45] and make the continuation to Euclidean signature)

$$W = -\frac{i}{(2\pi)^4 \cdot 3 \cdot 2^6} T_2 \int \mathcal{C}_3 \wedge \left(\operatorname{Tr} R^4 - \frac{1}{4}(\operatorname{Tr} R^2)^2\right) \ , \tag{5.7}$$

where T_2 is the membrane tension and $\mathcal{C}_3$ is 3-form potential, for the 11-dimensional action. This term was first derived in [46] and plays an important role in the inflow mechanism [47, 48]. Here we follow the line of reasoning suggested in [30]. We first integrate (5.7) by parts and then compactify on S^4 with flux[8] $T_2 \int_{S^4} F = 2\pi N$, $F = d\mathcal{C}_3$. The term (5.7) then reproduces exactly the Chern-Simons action to be added to the 7-dimensional gravitational action. This action is a source of the six-dimensional gravitational anomaly either through the holographic procedure or in the descent method analysis. The anomaly takes exactly the form conjectured in [44]. We can now determine explicitly values of the couplings β and β_1 in the Chern-Simons action. In the units in which radius of S^4 is $1/2$ we have that $16\pi G_N^{(7)} = 3\pi^3/N^3$ (see [45]) and hence $\beta = \frac{i}{2^9 N^2}$ and $\beta_1 = -\frac{i}{2^{11} N^2}$.

Acknowledgments

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[8]This is the right period for the form F, as was discussed in [46].

A. Curvature components

For metric (4.6) the components of the $(d+1)$-dimensional Riemann tensor are

$$R^r{}_{irj} = \frac{1}{2}\left[-g'' + \frac{1}{2}g'g^{-1}g'\right]_{ij}$$

$$R^r{}_{ikj} = -\frac{1}{2}\left[\nabla_k g'_{ij} - \nabla_j g'_{ik}\right]$$

$$R^l{}_{ikj} = R^l{}_{ikj}(g) - \frac{1}{4}g'_{ij}g^{ln}g'_{nk} + \frac{1}{4}g'_{ik}g^{ln}g'_{nj} \quad , \tag{A.1}$$

where $g' \equiv \partial_r g$. Components of Ricci tensor are

$$R_{ij} = R_{ij}(g) - \frac{1}{2}g''_{ij} - \frac{1}{4}g'_{ij}\operatorname{Tr}(g^{-1}g') + \frac{1}{2}(g'g^{-1}g')_{ij}$$

$$R_{ri} = \frac{1}{2}[\nabla_k(g^{-1}g')^k_i - \nabla_i\operatorname{Tr}(g^{-1}g')]$$

$$R_{rr} = -\frac{1}{2}\operatorname{Tr}(g^{-1}g'') + \frac{1}{4}\operatorname{Tr}(g^{-1}g'g^{-1}g') \tag{A.2}$$

and the Ricci scalar is

$$R = R(g) - \operatorname{Tr}(g^{-1}g'') - \frac{1}{4}[\operatorname{Tr}(g^{-1}g')]^2 + \frac{3}{4}\operatorname{Tr}(g^{-1}g'g^{-1}g') \quad . \tag{A.3}$$

B. Expansion for the inverse metric and the Riemann tensor

As preparation we present here expressions for the inverse of effective metric $g_{ij}(r,x)$ and its derivatives with respect to r

$$g^{-1} = e^{-2r}g_{(0)}^{-1}\Big[1 - g_{(2)}e^{-2r} + \Big(-g_{(4)} + g_{(2)}g_{(0)}^{-1}g_{(2)}\Big)e^{-4r}$$
$$+ \Big(-g_{(6)} + g_{(2)}g_{(0)}^{-1}g_{(4)} + g_{(4)}g_{(0)}^{-1}g_{(2)} - g_{(2)}g_{(0)}^{-1}g_{(2)}g_{(0)}^{-1}g_{(2)}\Big)e^{-6r} + \cdots\Big]g_{(0)}^{-1}$$
$$g' = 2e^{2r}\left(g_{(0)} - g_{(4)}e^{-4r} - 2g_{(6)}e^{-6r} + \cdots\right)$$
$$g'' = 4e^{2r}\left(g_{(0)} + g_{(4)}e^{-4r} + 4g_{(6)}e^{-6r} + \cdots\right) \quad , \tag{B.1}$$

where ... stands for the sub-leading terms. In particular we have that

$$g^{-1}g' = 2g_{(0)}^{-1}\Big\{1 - g_{(2)}e^{-2r} + \Big(-2g_{(4)} + g_{(2)}g_{(0)}^{-1}g_{(2)}\Big)e^{-4r}$$
$$+ \Big(-3g_{(6)} + 2g_{(2)}g_{(0)}^{-1}g_{(4)} + g_{(4)}g_{(0)}^{-1}g_{(2)} - g_{(2)}g_{(0)}^{-1}g_{(2)}g_{(0)}^{-1}g_{(2)}\Big)e^{-6r} + \cdots\Big\} \quad . \tag{B.2}$$

It is important to note that if the dimensions d is even there generally appears a logarithmic[9] term $h_{(d)}re^{-(d-2)r}$ in the expansion (4.7). In (B.2) this would add extra terms $g_{(0)}^{-1}(h_{(d)}e^{-dr} - (d-2)h_{(d)}re^{-dr})$ plus the corresponding higher order terms.

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[9]The logarithm appears if one uses the radial coordinate $\rho = e^{-2r}$ instead of r.

Using (A.1) we can get expansion for the components of the Riemann tensor. We find

$$R^r{}_{irj} = -\left(g_{(0)}e^{2r} + g_{(2)}\right)_{ij} + \cdots \tag{B.3}$$

$$R^{ri}{}_{rj} = -\delta^i_j + O(e^{-4r})$$

$$R^r{}_{ikj} = \left(\nabla_k g^{(2)ij} - \nabla_j g^{(2)ik}\right) + \cdots \tag{B.4}$$

$$R^{ri}{}_{kj} = \left(\nabla_k g^i_{(2)j} - \nabla_j g^i_{(2)k}\right) e^{-2r} + \cdots$$

$$R_{kj}{}^{ri} = \left(\nabla^k g^{(2)i}_j - \nabla^j g^{(2)i}_k\right) e^{-4r} + \cdots$$

$$R^l{}_{ikj} = \left(g_{(0)ik}\delta^l_j - g_{(0)ij}\delta^l_k\right) e^{2r} + \left(R^{(0)l}{}_{ikj} + g_{(0)ij}g^l_{(2)k} - g_{(0)ik}g^l_{(2)j}\right) + \cdots \tag{B.5}$$

$$R^{li}{}_{kj} = \left(\delta^i_k\delta^l_j - \delta^i_j\delta^l_k\right) + \left(R^{(0)li}{}_{kj} + \delta^i_j g^l_{(2)k} - \delta^i_k g^l_{(2)j} - \delta^l_j g^i_{(2)k} + \delta^l_k g^i_{(2)j}\right) e^{-2r} + \cdots$$

We use the inverse metric $g^{ij}_{(0)}$ to raise the indices. Useful expansion for the Levi-Civita symbol is

$$\epsilon^{i_1 \ldots i_d r} = e^{-dr}\epsilon^{i_1 \ldots i_d}_{(0)} + \cdots \quad, \tag{B.6}$$

where $\epsilon^{i_1 \ldots i_d}_{(0)}$ is defined with respect to the metric $g_{(0)ij}$.

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